Orthogonal Voronoi Molecules

Robby Kraft

Abstract: *This paper presents a method of constructing a Voronoi diagram in origami. The model presents a flagstone tiling of Voronoi cells in a flat plane with three dimensional pleats behind. Additionally, a folding sequence is offered which allows one to fold this algorithm without computer assistance.*

1 Introduction

The Voronoi diagram is well suited for origami design. The diagram's edges can be located using perpendicular bisectors, a construction known to origami artists as the origami axiom “fold a point to another point”. The convex-only polygon tilings in Voronoi diagrams is also a property of flat-folded origami crease patterns. Some artists have created origami representations of the Voronoi diagram [Mitani 16] [Gjerde 12]. Additionally, the Voronoi algorithm is an important computational step in the Origamizer algorithm [Demaine Tachi 17], a design which shares much in common with the design described in this paper.

The folded origami described here will appear to be a flagstone tiling from the front. Typically, origami flagstone tilings are flat models with pleats on the back connecting polygons between their vertices. The simplest flattening, folding to one side, requires one vertex’s pleats be either clockwise or counter-clockwise. This requires that adjacent vertices alternate their winding direction, not unlike a cycle of interlocked gears, which globally requires even-sided faces only. A Voronoi diagram being an irregular tiling cannot guarantee this for every case.

This paper describes a design which succeeds for all Voronoi diagrams by constructing three-dimensional molecules, setting pleats upright and orthogonal to the diagram plane, ensuring enough material is available at their intersections to be neatly resolved. The final model exhibits a flagstone tiling on one side, and on the other side three-dimensional pleats trace the edges of the Voronoi diagram.

1.1 Voronoi Diagrams

A two-dimensional Voronoi diagram is a space-partitioning algorithm that tiles space into convex polygons called *cells.* The input to the algorithm is a set of points called *sites,* and the algorithm places one cell around every site. The shape of the cells are defined in terms of distances to the sites: for any point in space, the site nearest to that point is the one contained within the same cell [de Berg 97]. Figure 1 is a Voronoi diagram.

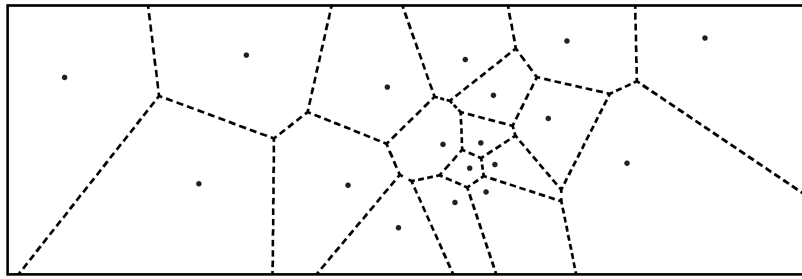


Figure 1: *a Voronoi diagram with sites (dots) and edges (dashed lines).*

Most geometry software today can compute Voronoi diagrams, it's easy to find one in your preferred language. Section 5.1 at the end of this paper outlines a non-rigorous approach to construct one by hand. However you choose to construct it, begin with a pre-computed 2D Voronoi diagram made from a set of sufficiently random sites.

1.2 Overview of the Crease Pattern

The Voronoi diagram edges are the first set of creases to be put in the crease pattern; they will all be the same crease assignment. In the upcoming section we will learn to construct cells and molecules, their crease assignments follow:

- The Voronoi edges, generated by the algorithm. (valley)
- The scaled Voronoi cells, generated by strip grafting. (mountain)
- The crease lines that compose the molecules. (valley and mountain)

2 Strip-Grafting the Voronoi Diagram

2.1 Cell transform

After the Voronoi diagram has been generated, the first step will be to separate the cells from one another. Notice in Figure 2, this spacing is of variable width.

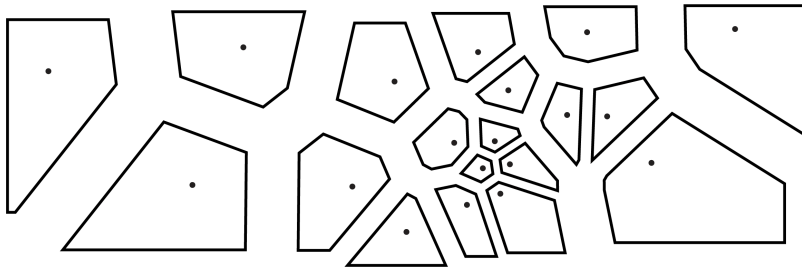


Figure 2: *Voronoi cells separate to prepare for strip-grafting.*

The technique of separating the cells is already familiar to origami designers as *strip grafting* [Lang 03], and can be imagined two ways: an additive process in which the artist cuts, separates, and pastes new material between the cells, or alternatively, shrinking the cells to reveal space between them; both perspectives generate the same result. In the case of our Voronoi-based design, the shrinking operation is an affine scale transform where the homothetic center is the cell's site. The scale acts upon every vertex **p** in the cell, translating it along a vector **v** from the cell's site **s** to the location of the vertex **p**. The new location **p′** is defined

$$\mathbf{v} = \overrightarrow{\mathbf{sp}} \tag{1}$$

$$\mathbf{p}' = \mathbf{s} + t\,\mathbf{v} \tag{2}$$

where

$$0 < t < 1 \tag{3}$$

t is the magnitude of the transformation. 0 and 1 are degenerate cases; this algorithm works for any value between the two. Whatever your choice is for t it should be applied globally. We will demonstrate later that when folding by hand it's best to select a value that is associated with bisections, like ½ or ¼.

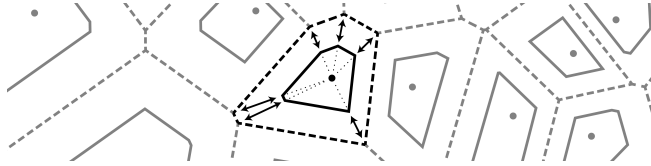
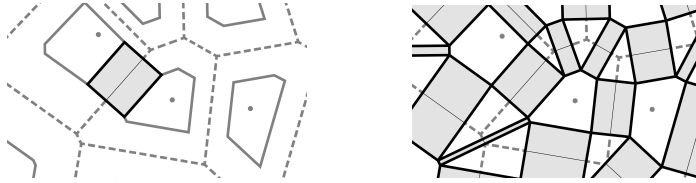


Figure 3: *Vertices move proportionally along the vector from the cell's site.*

2.2 Grafts

After the cells are scaled, what was originally one edge in the Voronoi graph has become two adjacent cell walls, each with the same length, and are reflections of one another across the Voronoi edge. If we connect the four vertices of this edge pair it forms a rectangle, shown in Figure 4 (a), where the Voronoi edge is an axis of symmetry. This rectangle is the *strip* in the strip grafting technique.



(a) **(b)**

Figure 4: *Strip grafting between two cells (a) and the entire graph (b).*

The dimensions l and w of the strip rectangle can be expressed in terms of the Voronoi edge **e** that lies under it and the two incident cell sites **q** and **p**:

$$l = t\|\mathbf{e}\| \tag{4}$$

$$w = u\|\mathbf{qp}\| \tag{5}$$

where u is the complement of t

$$u = 1 - t \tag{6}$$

When the origami is folded these strips act as pleats which move neighboring cells to become adjacent again. At this point, the origami deviates from most prior work, like in [Lang 15] [Demaine2 and Qaiser 14]. As this origami is not required to be flat, there are no limitations to the shape of cell polygons for example, and a solution is possible for a more general set of cases of Voronoi diagrams.

3 Voronoi Molecules

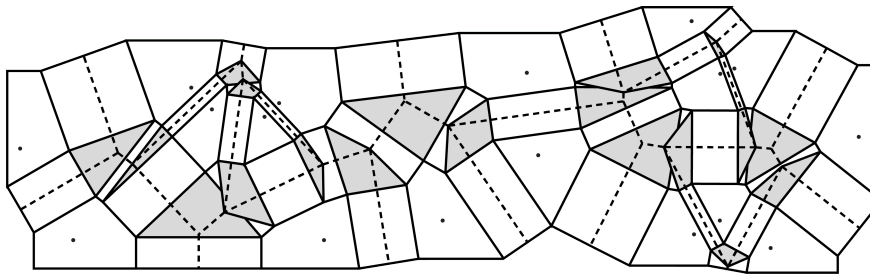


Figure 5: *the molecule areas are shaded.*

At this point it's possible to pleat the entire piece and get a neatly-presented flagstone tessellation of the Voronoi diagram on one side. However the backside will be a messy, unallocated bunching of paper at every pleat intersection, as shown in Figure 6 (a). The purpose of the molecules is to clean up this space.

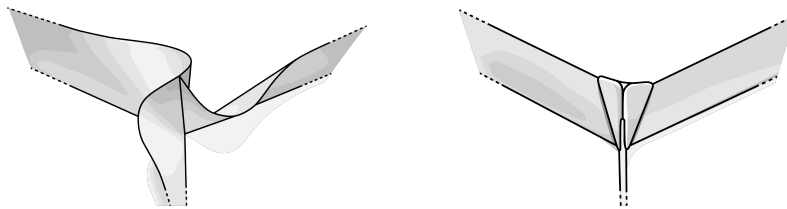


Figure 6: *Before (a) and after (b) applying the molecule.*

Molecules are located at the intersection of three strips or three cells, as shown in Figure 5. Molecules are the remaining areas left over that is not a cell or a strip; with one important addition: the molecule's triangle's circumcenter. The molecule is the convex hull of these four points, forming either a triangle or quadrilateral. Notice how some molecules in Figure 5 overlap each other. When this occurs one molecule's crease pattern will change according to a new set of rules described in Section 4. Before we worry about global interferences like this, let's focus on solving the crease patterns for an isolated molecule.

3.1 Circumcenters

Every molecule has an associated Voronoi vertex. The transform from Equation (1) is depicted again in Figure 7 (a). One vertex is copied and translated three times forming a triangle. Because each of the translations are of the same magnitude t from Equation (2), this Voronoi vertex is the triangle's circumcenter. If the triangle is obtuse, the circumcenter will lie outside the molecule triangle.

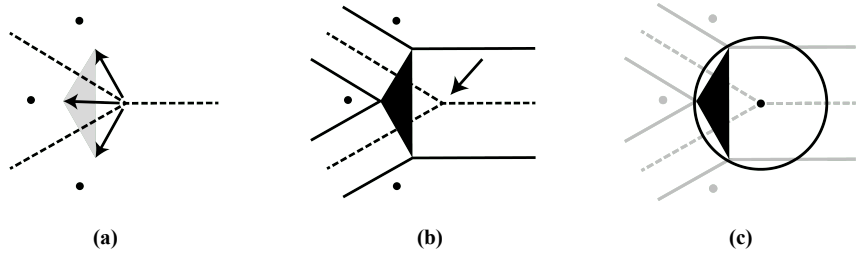


Figure 7: *A molecule is made from a Voronoi vertex, this molecule's circumcenter.*

We will begin by solving each molecule's crease pattern on an individual basis; treating each according to the type of triangle: acute, right, and obtuse.

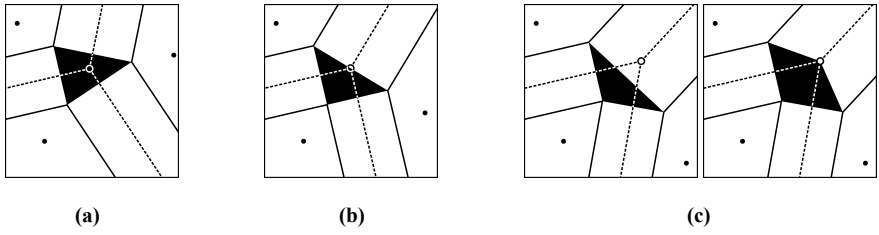


Figure 8: *Acute (a), right (b), and obtuse (c) without and with its circumcenter.*

3.2 Acute Molecules

The crease pattern for any molecule is solved by imagining a molecule divided into smaller triangles. Figure 9 (a) shows a molecule's triangle space. Initially, this only contains the Voronoi edges from the Voronoi digram. These Voronoi

edges intersect at the triangle's circumcenter. Figure 9 (b) adds circumradii, one line from the circumcenter to each of the three vertices. These lines will used to imagine separating the molecule into smaller isosceles triangles as in Figure 9 (c).

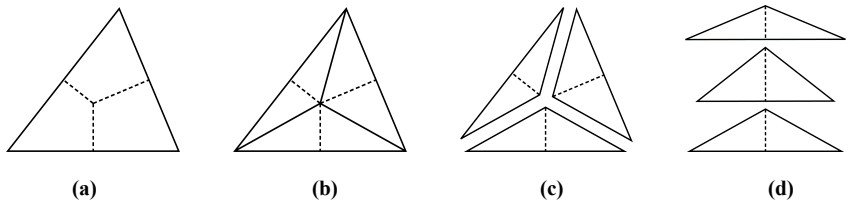


Figure 9: *Divide a triangle by its circumradii into smaller isosceles triangles.*

Each of these isosceles triangles will become rabbit ear molecules. This requires two additional creases formed by bisecting the base angles, meeting at the triangle's incenter, shown in Figure 10 (a). Finally, the perpendicular that descends to the base side of the triangle should reverse its crease assignment.

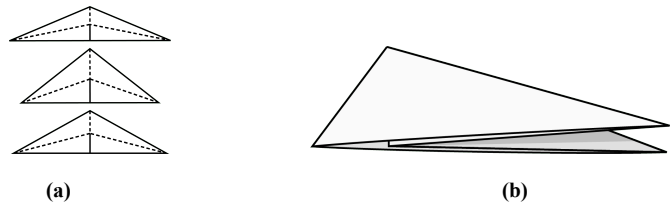


Figure 10: *Rabbit ear crease patterns (a) and example folded (b).*

Moving these isosceles triangles back into place finishes the molecule for the acute triangle case, shown in Figure 11. Figure 6 (b) is the folded form.

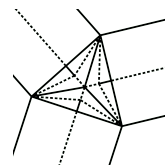


Figure 11: *the finished crease pattern for an acute molecule.*

3.3 Right Molecules

Equipped with our isosceles rabbit ear technique let's look back on the three types of molecules. For our purposes, we say the right triangle contains only two isosceles units, ignoring the third degenerate triangle unit, shown in Figure 12 (b). The obtuse case in Figure 12 (c) appears to have three isosceles units again, but

one of them is inverted, it has an opposite winding, and will serve a unique purpose in the obtuse molecule construction.

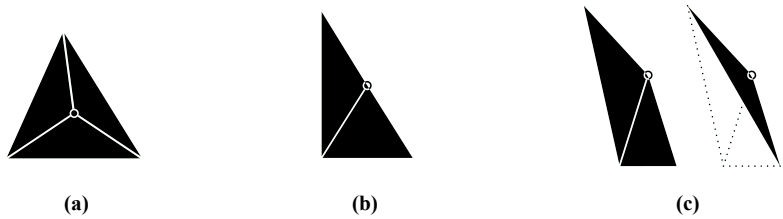


Figure 12: *Isosceles triangles inside acute (a) right (b) and obtuse (c) molecules.*

The right triangle molecule is solved by applying the rabbit ear technique to the two available isosceles triangles; otherwise following the exact same process for the acute molecule. This completes the right triangle molecule.

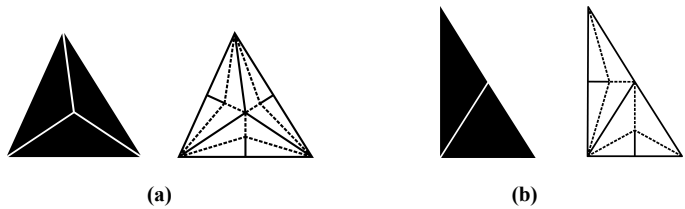


Figure 13: *Acute (a) and right (b) molecules with their arrangement of crimps.*

3.4 Obtuse Molecules

Similarly with the right triangle molecule, the obtuse molecule results in two isosceles triangle rabbit ear constructions; formed from the two isosceles units with the same winding. However, unlike both acute and right molecules, the obtuse molecule's rabbit ears's angles are not simple bisections. Instead, the two isosceles units are transformed into smaller isosceles triangles, then the simple rabbit ear bisection is applied to these. To calculate the smaller triangles we need the third isosceles triangle, the one with the opposite winding from Figure 12 (c). For this calculation we simply need to focus on the base angles of each triangle. Let α and β be the base angles for the two visible isosceles triangles and γ be the base angle of the third triangle with the opposite winding. We will form two new isosceles triangles with base angles α' and β' such that,

$$\alpha' = \alpha - \gamma$$

$$\beta' = \beta - \gamma \tag{7}$$

Figure 14 visually demonstrates this process. Because the third triangle unit lies inside the other two, the subtraction in Equation (7) can be done geometrically by

marking the third triangle's base edge onto the other two triangles, as shown in Figure 14 (c). Figure 14 (d) constructs two new isosceles triangles using this new line by replacing the middle portion between the two medians with two lines to the shared vertex. Finally in Figure 14 (e), rabbit ears are applied to these new isosceles triangles as was done in both the acute and right molecules.

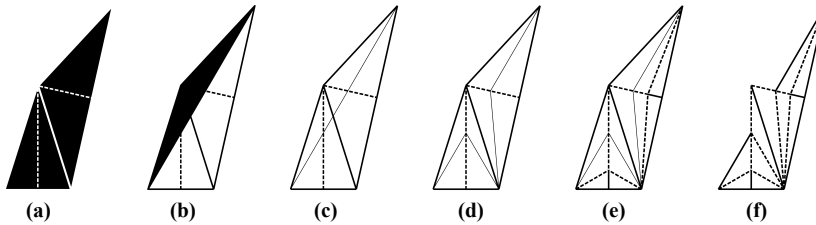


Figure 14: *Calculating the crease lines in an obtuse molecule*

Figure 14 (f) shows the final crease assignment. The outside edges of the original isosceles triangle pair are removed, but the shared edge between them remains. The creases in the rabbit ear units are consistent with the acute and right molecules, but the outer top edges of these two isosceles units alternate assignment, mirroring one another. Figure 15 demonstrates one such folding, the area between the two indicators in Figure 15 (b) is coplanar and flat.

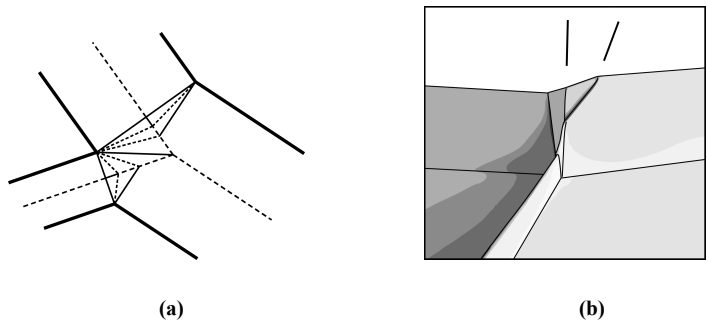


Figure 15: *The crease pattern for an obtuse molecule (a) and its folded form (b).*

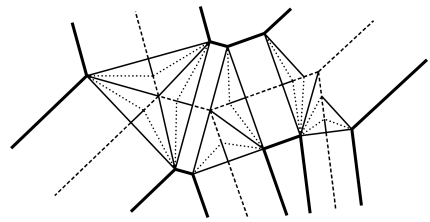


Figure 16: *Three non-overlapping molecules, left to right: acute, right, obtuse*

4 Overlapping Voronoi Molecules

Because an obtuse molecule's circumcenter exists outside its triangle it's possible that two molecules are so close that one circumcenter is contained inside another molecule as shown in Figure 17. We call these molecules overlapping.

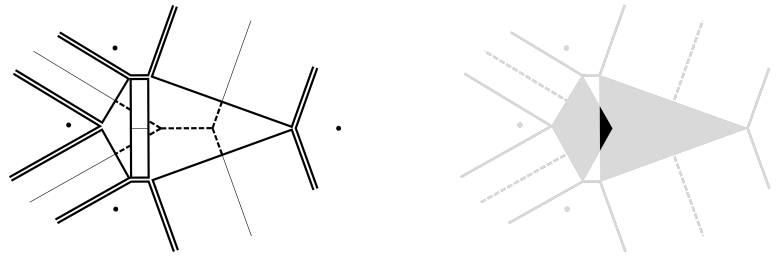


Figure 17: *Overlapping molecules.*

A molecule must be obtuse if it occupies another molecule's space, but an occupied molecule can be any of the three types. When two molecules overlap only one of the molecule's crease patterns will be modified; the altered molecule is the one being occupied by the other's circumcenter. In both Figure 17 and 18, the molecule on the right's crease pattern will be changed.

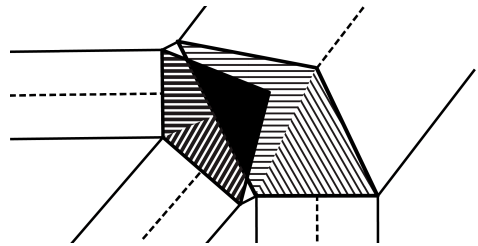


Figure 18: *Two overlapping obtuse molecules with their isosceles units shaded.*

By hatch-filling, Figure 18 is showing the isosceles units that compose each molecule. Each of these molecules are obtuse and have two isosceles units. Notice in the occupied molecule on the right, only one isosceles units is being overlapped. It will always be the case that between two overlapping molecules, only one isosceles unit is occupied in the occupied molecule.

4.1 Overlapping Molecules' Crease Patterns

Between two overlapping molecules, most of the crease pattern is still able to be solved by the process from Section 3. The areas which should be avoided until now are any isosceles units occupied another molecule's circumcenter. This is marked by an empty triangle in Figure 19 (a); this is the space where one rabbit

ear crimp would have be placed. Broadly, our goal remains the same, to fit one rabbit ear crimp here, but it will no longer be a simple crimp.

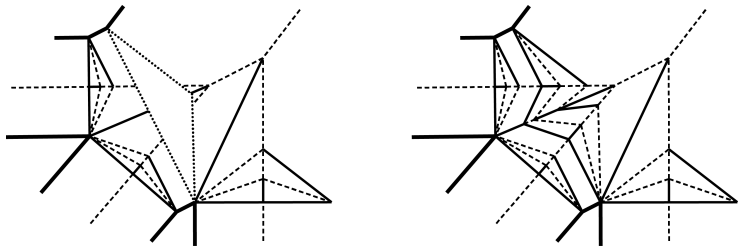


Figure 19: *The overlapped isosceles unit unsolved (a) and solved (b).*

The process for filing this overlapped isosceles unit is a stepwise one, working inwards from either of the two isosceles base vertices. Begin at each corner by applying the same rabbit ear crimp, casting crease lines like rays from the vertex. Whenever these rays intersect a line, reflect across the line and continue propagation, and when repeated, both sets of rays meet in the middle. Propagation can begin from either end and all lines will meet at the opposite vertex.

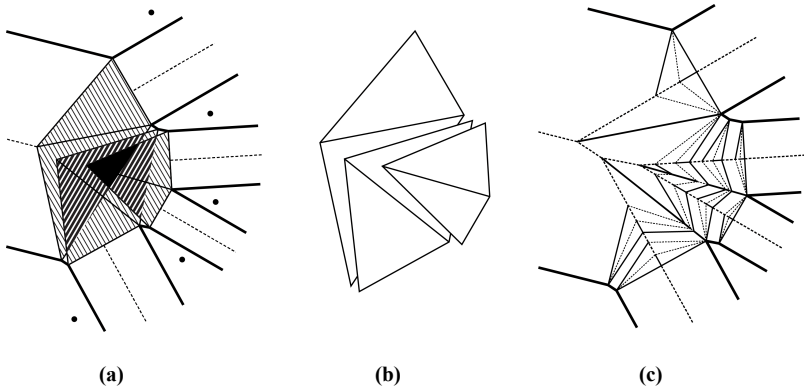


Figure 20: *Three overlapping molecules (a) the black area is twice overlapped (b) detail of the overlapping arrangement, and (c) the final crease pattern.*

It is possible for one molecule to be overlapped by many, not just one. However, the process for solving the crease pattern remains the same. For every molecule, gather a list of other molecules which occupy this molecule; the lines from these molecules are the lines to test for intersections and reflect across while generating the rabbit ear crimps. Given this set of lines, the processes is the same as described above. Figure 20 illustrates multiple overlaps.

5 Folding Voronoi Molecules

As it turns out the origami introduced in this paper is able to be folded by hand, so I wanted to take advantage of this in the final section. This paper above skips over the Voronoi algorithm itself, and you can too if you choose to print out a Voronoi diagram, skip to Section 5.2. However I've learned to fold the Voronoi diagram itself by hand, which draws upon a new skillset unfamiliar to seasoned origami artists. I describe this non-rigorous, intuition building process in Section 5.1. Either approach requires beginning with a set of sufficiently randomly placed points on the page. I typically crease a Voronoi origami four consecutive steps:

1. Crease the lines of the Voronoi algorithm.
2. Crease the outlines of the scaled cells.
3. Crease the Kawasaki edge lines.
4. Crease the insides of the Voronoi Molecules.

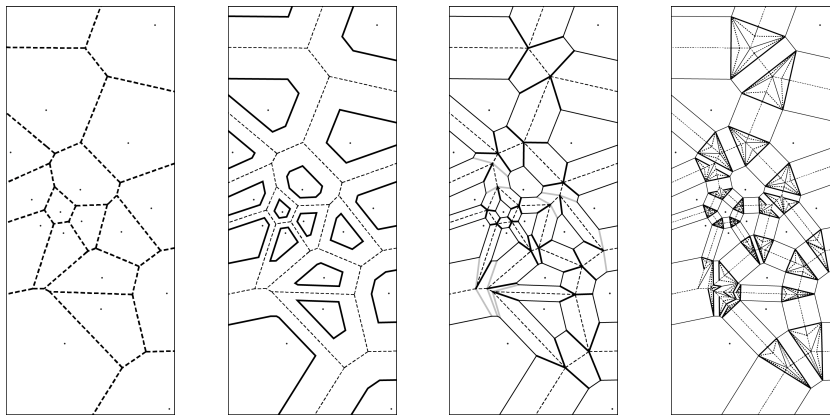


Figure 21: *(1) Voronoi, (2) cells, (3) Kawasaki, (4) molecules*

5.1 Folding a Voronoi Diagram by Hand

In this section we will learn how to uncover a Voronoi diagram by hand. I encourage newcomers to use the information in this section to create drawings of Voronoi diagrams as a reinforcement to learning. Plus, the sketches end up being works of art on their own.

Folding a Voronoi diagram requires repeating only one origami axiom instruction, "make a crease by bringing one point onto another point." The trick here is to learn which set of points to fold and how long each crease line should be. The instruction I repeat over and over is:

Locate the two points which are globally nearest to each other and crease between them only a small portion of an edge.

Repeat that step, each time moving to the next globally-nearest pair. This loop should be run in parallel with the instruction:

When an intersection between three creases is apparent, extend the edges to meet.

Figure 22: *These three sites form an acute triangle. The intersection of their Voronoi edges is readily apparent.*

There is one important caveat to this loop. While all pairs of neighboring Voronoi sites have a perpendicular bisector line, the segment formed along the line does not always include the portion directly between the two sites. When this occurs these are the obtuse triangle molecules, shown in Figure 23. After creasing the two closest pairs in Figure 23 (a), it becomes apparent that the intersection point lies outside of the boundary of an imaginary triangle formed between the points. The consequence is that the crease line between the furthest two sites doesn't occupy the space directly between them.

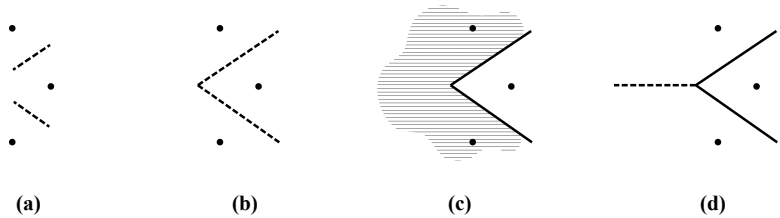


Figure 23: *The sequence to crease between 3 sites forming an obtuse triangle.*

Figure 23 (c) shows the shaded area "owned" by these two sites. This is the essence of the Voronoi algorithm. Folding this enough times will give you an intuition about space owned by points and the ability to anticipate where to stop a crease. Figure 24 shows an exaggerated case of an obtuse triangle arrangement.

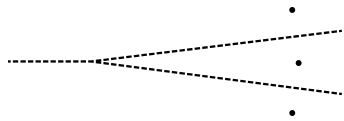


Figure 24: *As the obtuse triangle arrangement approaches the degenerate case, the center cell rapidly elongates*

To master this, one exercise is to draw two clusters of points at opposite ends of a space, where these clusters cast lines generally in the direction of the other. The challenge is to successfully resolve the creases between the sets of points.

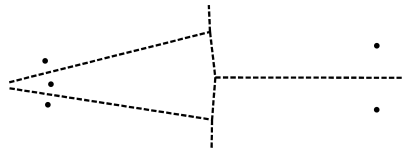


Figure 25: *A challenging exercise: a pair of clusters separated by a large distance*

Here is another challenge: when four points nearly take the shape of a square, or any number forms a regular polygon, more than three edges will appear to intersect at the same point. It's a challenge to tell which three lines intersect first.

5.2 Folding the Strip Grafts

Strip grafting cells is easily accomplished using the scaling method from Section 2, the simplest value for t in Equation (3) for folding is ½. This places the Voronoi cell edge crease halfway between the site and the Voronoi edge.

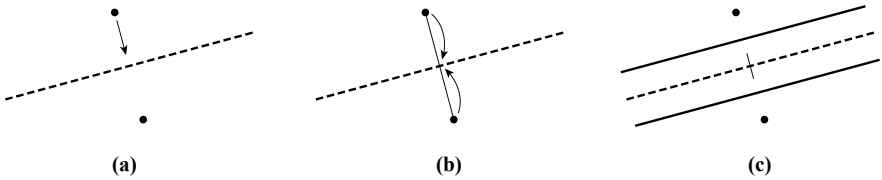


Figure 26: *Folding sequence for strip grafting and shrinking cells.*

A precise folding requires one pre-crease followed by two folds. The pre-crease is made by the axiom "make a crease that passes through two points". However, a small mark at the midpoint is all that is needed, shown in Figure 26 (c). After folding, this mark will be hidden inside the pleat. Next, make two creases by folding the pair of points to the mark at their midpoint. These new creases should be the opposite crease assignment as the Voronoi edge creases.
The precise length of these creases will be unknown until the neighbors on each side are creased too. It's possible to fold short segments for every side of a polygon, then lengthen each crease until the polygon's edges are connected.

5.3 Kawasaki's Theorem

This step creases the circumradii lines discussed in Section 3.2. As it turns out, these are also constructed using a convenient origami technique.

Satisfying Kawasaki's theorem is one of the necessary conditions for single-vertex flat-foldability [Hull 94]. Ignoring crease assignment for now, given three creases in a single-vertex, Kawasaki's theorem can be used to calculate a fourth crease which will make the unit flat-foldable. In the case that a degree-3 single vertex's sector angles are all less than 180°, Kawasaki's theorem will uncover three potential solutions to satisfy flat-foldability, one inside of each sector.
In our Voronoi diagram design, when build from sufficiently random sites, every intersection of edges exhibits these properties:

- Each intersection contains three creases.
- Because every cell is convex there are no sector angles < 180°.
- All three edges at a Voronoi intersection are the same crease assignment.

A single vertex of three similarly-assigned creases with three sector angles each smaller than 180° can be collapsed by hand and a fourth crease will be discovered automatically as the necessary conditions of flat-foldability are self-satisfied. Once the single vertex is flat, the flaps can be toggled to uncover the two remaining flat folded states, adding two additional creases.
An interesting consequence of doing this on a Voronoi vertex is that the Kawasaki solutions pass through each of the three Voronoi cell sites.

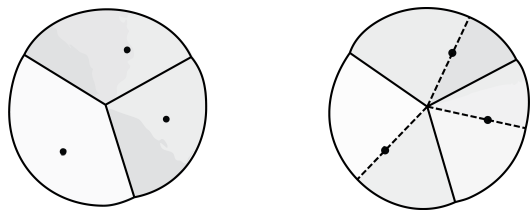


Figure 27: *Kawasaki solutions on a Voronoi vertex are collinear with cell sites*

Revisit Figure 14 (f); when folding obtuse molecules, two Kawasaki lines are unnecessary. In Section 2.1 we defined *t*, the magnitude of the transformation, again mentioned in Section 5.2. Whenever I fold by hand, I set *t* to ½, making Kawasaki creases extend halfway to the sites, ending at the scaled cell vertices.

5.4 Folding the Molecules

As mentioned in Figure 6 (a), it's possible to collapse the origami without folding the molecules. This constructs a folded Voronoi that will appear finished from its front side. Additionally, the pleated twists lock the paper in a different manner. Before creasing the molecules, it's worth collapsing at this stage to take notice.
To crease the molecules use Section 4 as a reference. Overlapping molecules can be challenging for a beginner, avoid them until you are ready. You will discover this piece is not rigidly-foldable. Collapsing is easiest when starting from the center, the boundary areas are the simplest so save them for the end.

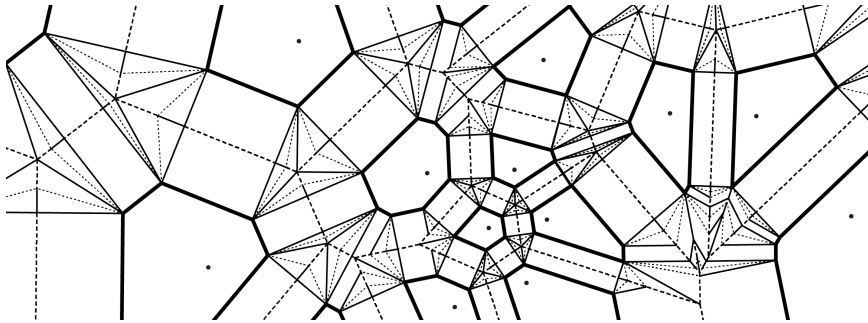


Figure 28: *A completed crease pattern*

References

[Hull 94] Tom Hull. “On the Mathematics of Flat Origamis.” Congressus Numerantium, Vol. 100 (1994), 215-224.

[de Berg 97] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars. *Computational Geometry*. Third Edition. Springer, 1997, 2000, 2008.

[Lang 03] Robert J. Lang. *Origami Design Secrets: Mathematical Methods for an Ancient Art*. Natick, MA: A K Peters, Ltd., 2003.

[Gjerde 12] Eric Gjerde. “Holland Paper Biennale 2012.” Origami Tessellations. www.origamitessellations.com/2012/08/holland-paper-biennale-2012/. 2012.

[Demaine² and Qaiser 14] Erik D. Demaine, Martin L. Demaine, and Kayhan F. Qaiser, “Scaling a Surface down to Any Fraction by Twist Folding”, in Origami6: Proceedings of the 6th International Meeting on Origami in Science, Mathematics and Education (OSME 2014), volume 1, Tokyo, Japan, August 10–13, 2014, pages 201–208, American Mathematical Society.

[Lang 15] Robert J. Lang “Spiderwebs, Tilings, and Flagstone Tessellations.” Koryo Miura, Toshikazu Kawasaki, Tomohiro Tachi, Ryuhei Uehara, Robert J. Lang, and Patsy Wang-Iverson, editors. Origami 6, I: Mathematics. American Mathematical Society, 2015.

[Mitani 16] Jun Mitani, *3D Origami Art*, A K Peters/CRC Press, June 2016.

[Demaine and Tachi 17] Erik D. Demaine and Tomohiro Tachi, “Origamizer: A Practical Algorithm for Folding Any Polyhedron”, in *Proceedings of the 33rd International Symposium on Computational Geometry (SoCG 2017)*, Brisbane, Australia, July 4–7, 2017, 34:1–34:15.

An interactive implementation is at: robbykraft.com/origamivoronoi/

Robby Kraft
School for Poetic Computation, 155 Bank St. New York City, New York 10014, e-mail: robbykraft@gmail.com